INDECOMPOSABLE HIGHER CHOW CYCLES ON JACOBIANS

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ABSTRACT. We construct some natural cycles with trivial regulator in the higher Chow groups of Jacobians. For hyperelliptic curves we use a criterion due to J. Lewis to prove that the cycles we construct are indecomposable, and then use a specialization argument to prove indecomposability for more general curves.

0. INTRODUCTION

The aim of this paper is to construct some natural cycles in the higher Chow groups of Jacobians of smooth projective curves. Most of the paper is devoted to the first higher Chow groups $CH^k(J(C), 1)$, especially the case $k = g = genus(C)$, but in the last section we also construct elements of $CH^k(J(C), n)$, $n > 1$, for curves of low genus.

For a smooth projective variety X, the subgroup of $CH^k(X, 1)$ of decomposable cycles, $CH^k_{\text{dec}}(X, 1)$, is defined to be the image of $CH^{k-1}(X) \otimes \mathbb{C}^*$, where we use the isomorphism $CH^1(X, 1) \cong \mathbb{C}^*$. We let $CH^k_{\text{ind}}(X, 1) := CH^k(X, 1)/CH^k_{\text{dec}}(X, 1)$ be the quotient group of indecomposable cycles. In [4], a natural element K was constructed in $CH^g(J(C), 1)$, C hyperelliptic, and using the regulator map K was shown to give a non-torsion element of $CH^g_{\text{ind}}(J(C), 1)$ when C is generic hyperelliptic. The Pontryagin product of K with zero cycles of degree zero, gives elements of $CH^g(J(C), 1)$ which lie in the kernel of the regulator map. Our first result, Theorem 1.2, shows that such cycles give uncountably many elements of $CH^g_{\text{ind}}(J(C), 1)_{\mathbb{Q}}$ if $g \geq 3$. We prove a more precise statement using the decomposition of the higher Chow groups due to Beauville [1] and Deninger-Murre [5]. Applying the motivic hard Lefschetz theorem of Künnemann [11] we obtain uncountably many elements of $CH^k_{\text{ind}}(J(C), 1)_{\mathbb{Q}}$, $3 \leq k \leq g$, lying in the kernel of the regulator map. The main technical tool used in the proof is a Hodge theoretic criterion due to J. Lewis [12]. This has been used earlier by Gordon and Lewis [9] to construct indecomposable cycles with similar properties in products of generic elliptic curves.

Our main goal is to construct indecomposable cycles in $CH^g(J(C), 1)$ for more general curves. A result of Nori [14, 7.5] implies that upto torsion the regulator image of $CH^g(J(C), 1)$ is the same as that of $CH^g_{\text{dec}}(J(C), 1)$ for a generic curve of genus $g = 3$, and it seems likely that this should also be true for higher g. This makes it difficult to use Lewis' criterion to prove indecomposability; instead we employ a specialization argument. To this end, we first prove that the specialization of a decomposable cycle is decomposable (Theorem 2.1). The difficulty here is that

a cycle on a family of smooth projective varieties which restricts to a decomposable cycle on a generic fibre need not be decomposable on the entire family. We circumvent this problem by considering an auxiliary family which is constructed by gluing a product family along a special fibre.

Given a divisor $D = (a_1 + a_2) - (b_1 + b_2)$ on a curve C of genus $g \geq 2$ with $2[D] = 0$ in $J(C)$, we construct an element Z_D of $CH^g(J(C), 1)$, called the 4-configuration since it is supported on 4 copies of C in $J(C)$. When C and D are generic and $g \geq 3$, we show (Theorem 3.1 that this element is indecomposable: By using a suitable Hurwitz scheme, we show that we may specialize C to a hyperelliptic curve C' in such a way that the 4-configuration specializes to $K - K_t$, with t a generic point of C'. Indecomposability follows by applying Theorem 1.2 and 2.1. We then deduce that $CH^g_{\mathrm{ind}}(J(C), 1)_{\mathbb{Q}}$ is uncountable for generic curves of genus 3 and 4.

To construct elements of $CH^g(J(C), 1)$ for arbitrary curves, we consider divisors $D = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$ with $2[D] = 0$ in $J(C)$ and $n > 0$. Associated to such a divisor there is a natural subspace of $CH^{n+1}(C^{n+1}, 1)_{\mathbb{Q}}$ and we describe a method, generalizing that used for the 4-configuration, which we believe should enable one to prove indecomposability of the general such element with C and D also generic and $g \geq n + 1$. Unfortunately, due to the combinatorial complexity of the cycles involved, we have not been able to complete the proof. However, we have checked indecomposability using a simple computer program for $n \leq 6$. In particular, we see that $CH^g_{\mathrm{ind}}(J(C), 1)_{\mathbb{Q}}$ is uncountable for a generic curve with $3 \leq genus(C) \leq 12$.

We conclude the paper with a construction of elements in $CH^3(J(C), 4 - g)$, with $g \leq 2$. These may be viewed as successive degenerations of the 4-configuration on a genus 3 curve, as the curves acquire nodes. We expect, but do not prove, that these elements are indecomposable in a sense stronger than that of Lewis. We do prove however, using Lewis's criterion, that for B a bielliptic genus 2 curve $CH^3(J(B), 2)/Im(K_2(\mathbb{C}) \otimes CH^1(J(C)))$ is non-trivial modulo torsion.

Conventions. All varieties will be over the complex numbers $\mathbb{C}$ and all points will be closed points. We shall say that a condition holds for a *general* point of a variety if it holds for all points in a Zariski open subset and it holds for a *generic* point if it holds for all points outside a countable union of proper subvarieties.

We denote by $*$ the Pontryagin product on the higher Chow groups of an abelian variety.

1. Lewis' conditions, Pontryagin products and hyperelliptic Jacobians

The first higher Chow group $CH^k(X, 1) \simeq H^{k-1}(X, \mathcal{K}_k)$ of a non singular variety X is generated by higher cycles of the form $Z = \sum_i Z_i \otimes f_i$, where the Z_i are irreducible subvarieties of codimension $(k - 1)$ and the rational functions $f_i \in k(Z_i)^\times$ obey the rule $\sum_i div(f_i) = 0$ as a cycle on X. Consider the subgroup of *decomposable* cycles

$$(1.1) \qquad CH^k_{\mathrm{dec}}(X, 1) := Im\{CH^1(X, 1) \otimes CH^{k-1}(X) \longrightarrow CH^k(X, 1)\} ,$$

and the related quotient of *indecomposable* cycles

$$CH^k_{\mathrm{ind}}(X, 1) := CH^k(X, 1) / CH^k_{\mathrm{dec}}(X, 1) .$$

Recall that if X is projective, then $CH^1(X, 1) = \mathbb{C}^*$.

Let I be the zero cycles of degree zero on an abelian variety A. Bloch has shown that I^{*n} is non-zero for $1 \leq n \leq g$ whereas $I^{*(g+1)}$ is always zero, $g = dim(A)$. Given Z an indecomposable element in $CH^k(A,1)$, a natural question to ask is whether $I^{*n} * Z$ contains indecomposable cycles for $n \geq 1$. We remark that $I^{*n} * Z$ is in the kernel of the regulator map for any Z, since translation acts trivially on the Deligne cohomology $H^{2k-1}_{\mathcal{D}}(X, \mathbb{Z}(k))$.

Using as a basic tool a condition of Hodge type due to Lewis, for Z a real regulator indecomposable element of $CH^g(J(C),1)$ we give a criterion in terms of the primitive cohomology of $J(C)$ for $I^{*n} * Z$ to contain indecomposable cycles. A significant instance of this situation is the case of generic hyperelliptic Jacobians, where we can use as Z the basic cycle in $CH^g_{\mathrm{ind}}(J(C),1)$ found in [4]. We show that $I^{*n} * Z$ contains indecomposable cycles for $1 \leq n \leq g-2$, whereas all elements of $I^{*(g-1)} * Z$ are decomposable (with $\mathbb{Q}$ coefficients).

1.1. Preliminaries.

We recall some notation and definitions and then we state a theorem of Lewis. Our aim is to have a concrete reference at hand, for more details the reader should consult either the original paper [12] or the survey [10].

For X projective and nonsingular,

$$\frac{H^{i-1}(X, \mathbb{C})}{F^j H^{i-1}(X, \mathbb{C}) + H^{i-1}(X, \mathbb{R}(j))} \simeq \frac{H^{i-1}(X, \mathbb{R}(j-1))}{\pi_{j-1}(F^j H^{i-1}(X, \mathbb{C}))}$$

and therefore one has the identification

$$H^{2k-1}_{\mathcal{D}}(X, \mathbb{R}(k)) \simeq H^{2k-2}(X, \mathbb{R}(k-1)) \cap F^{k-1} H^{2k-2}(X, \mathbb{C})$$
$$=: H^{k-1,k-1}(X, \mathbb{R}(k-1)).$$

According to Beilinson [2], the real regulator image of a cycle $Z = \sum Z_i \otimes f_i$ in $CH^k(X,1)$ is the element

$$R_{k,1}(Z) \in H^{2k-1}_{\mathcal{D}}(X, \mathbb{R}(k)) \simeq H^{k-1,k-1}(X, \mathbb{R}(k-1)),$$

determined by the class of the current

$$R_{k,1}(Z) : \omega \longmapsto (2\pi\sqrt{-1})^{k-1-d} \sum_i \int_{Z_i - Z_i^{\mathrm{sing}}} \omega \, \log |f_i| \,.$$

Definition. A higher Chow cycle $Z \in CH^k(X,1)$ is said to be (real) *regulator indecomposable* if there exists a differential form

$$\omega \in (Hdg^{k-1}(X) \otimes \mathbb{R})^{\perp} \subset H^{d-k+1,d-k+1}(X, \mathbb{R}(d-k+1))$$

such that the pairing $[R_{k,1}(Z), \omega] \neq 0$.

Lemma 1.1. *If $Z \in CH^k(X,1)$ is regulator indecomposable then Z is indecomposable.*

Recall that the *coniveau filtration* on $H^i(X, \mathbb{Q})$ is

$$N^j H^i(X, \mathbb{Q}) := \ker \left(H^i(X, \mathbb{Q}) \longrightarrow lim_{codim_X Y \geq j} H^i(X - Y, \mathbb{Q}) \right),$$

where the direct limit is over closed subvarieties $Y \subset X$. The complex subspace generated by the Hodge projected image of the coniveau filtration is

$$H^{k-l,k-m}_N(X) := Im(N^{k-l} H^{2k-l-m}(X, \mathbb{Q}) \otimes \mathbb{C} \longrightarrow H^{k-l,k-m}(X)) \,.$$

Lewis constructs certain complex subspaces

$$H^{\{k,l,m\}}(X) \subseteq H^{k-l,k-m}(X),$$

such that for $m = 0$ one has

(1.2) $$H^{\{k,l,0\}}(X) \subseteq H_N^{k-l,k}(X).$$

The spaces $H^{\{k,l,m\}}(X)$ are obtained by a process of Künneth projection of the Hodge components of the real regulator classes of the elements in $CH^k(X \times S, m)$, with varying smooth projective varieties S.

We give an abridged version of the main result from [12], since we only need it for $CH_{\mathrm{ind}}^k(X, 1)$.

Theorem 1.1 (Lewis). *Let X be a non singular projective variety:*

(i) $H^{\{k-1,l-1,0\}}(X) \subset H^{\{k,l,1\}}(X)$.

(ii) *If $H^{\{k,l,1\}}(X)/H^{\{k-1,l-1,0\}}(X) \neq 0$ for some l, $2 \leq l \leq k$, then $CH_{\mathrm{ind}}^k(X, 1)_{\mathbb{Q}}$ is uncountable.*

Note that Lewis' proof shows that for a cycle $\xi \in CH^k(X \times S, 1)$ which gives rise to a nonzero element of $H^{\{k,l,1\}}(X)/H^{\{k-1,l-1,0\}}(X)$, the cycles ξ_s, $s \in S$, form an uncountable subset of $CH_{\mathrm{ind}}^k(X, 1)_{\mathbb{Q}}$.

Our results will show that (ii) holds for a generic hyperelliptic Jacobian of genus $g \geq 3$, for $k = g$ and all l such that $2 \leq l \leq g - 1$.

1.2. **Lewis' condition holds for the Pontryagin families $Z(m)$.** Let C be a smooth projective curve of genus g and $J = J(C)$ be its Jacobian. Let $Z \in CH^g(J(C), 1)$ be a regulator indecomposable cycle. Given m points p_i on C, we construct higher cycles $Z(m)$ in $CH^g(C^m \times J(C), 1)$, so that the fibre over $(t_1, \ldots, t_m)$ is $Z * ([t_1 - p_1] - e) * \cdots * ([t_m - p_m] - e)$, where e is the origin in $J(C)$. Set $Z(1) := b_*(C \times Z) - C \times Z$, where $b : C \times J(C) \to C \times J(C)$ is the twisted isomorphism $b(t, x) = (t, [t - p_1] + x)$. By iteration this gives $Z(m)$ on $C^m \times J(C) = C \times (C^{m-1} \times J(C))$, where the twisted map is now $b((t_m, \ldots, t_1, x)) = (t_m, \ldots, t_1, [t_m - p_m] + x)$.

Proposition 1.1. *Let C be a smooth projective curve of genus g and Z a regulator indecomposable cycle in $CH^g(J(C), 1)$. Assume that the primitive cohomology $P^i(J(C))$ is an irreducible sub-Hodge structure of $H^i(J(C), \mathbb{Q})$ for some $m \leq g - 2$ and all $i \leq m + 2$. Then the real regulator of $Z(m)$ gives rise to a non-zero element of $H^{\{g,m+1,1\}}(J(C))/H^{\{g-1,l,0\}}(J(C))$, and hence the restriction of $Z(m)$ to the fibre over a generic point of C^m is indecomposable.*

If $\omega_1^J, \ldots, \omega_g^J$ is a basis for $H^0(J(C), \Omega_J^1(C))$ such that the restriction $\zeta_i := \omega_{i|C}^J$, $1 \leq i \leq g$, is an orthonormal frame for $H^0(C, \Omega_C^1)$, then the class of the divisor Θ is determined by the form $\theta_J(C) = (i/2) \sum_{j=1}^g \omega_j^J \wedge \bar{\omega}_j^J$. We define $\tau_C = \omega_1^J \wedge \bar{\omega}_1^J - \omega_2^J \wedge \bar{\omega}_2^J$.

Lemma 1.2. *Under the assumptions of Proposition 1.1, if Z is real regulator indecomposable then there is a basis as above with $[R(Z), \tau] \neq 0$.*

Proof. The inner product on $H^0(C, \Omega_C^1)$ allows us to view it as a representation V of the unitary group $U(g)$. $H^{1,1}(J(C))$ is then also a representation of $U(g)$, and is isomorphic to a twist of $V \otimes V^*$ by a 1-dimensional representation. Hence by Pieri's

formula we see that it decomposes as a direct sum of two irreducible representations T and U, T being the subspace corresponding to the class of the Θ divisor.

Now consider the subspace W of $H^{1,1}(J(C))$ spanned by all possible τ's as above. Since the unitary group preserves the inner product, it follows that W is a subrepresentation of $H^{1,1}(J(C))$. Clearly W is not equal to T, hence it must contain U. If there were no τ's with a non-zero pairing with Z then the pairing would be zero on all of W, contradicting the assumption of real regulator indecomposable. (Note that our assumptions imply that the space spanned by the rational Hodge classes is T). □

Let $\alpha_m = \bar{\zeta}_{m+2} \ldots \bar{\zeta}_3 \wedge \tau \wedge \omega_3^J \wedge \ldots \omega_{m+2}^J$.

Lemma 1.3. $[R(Z(m)), \alpha_m] = (-1)^m [R(Z), \tau]$

Proof. By iteration the proof is the same for all $m \geq 1$. Say $m = 1$, then we have $b^*(\omega_s^J) = \omega_s^J + \zeta_s$ and $b^*(\zeta_s) = \zeta_s$, thus

$$b^*(\bar{\zeta}_3 \wedge \tau \wedge \omega_3^J) = \bar{\zeta}_3 \wedge b^*(\tau \wedge \omega_3^J) = -(\zeta_3 \wedge \bar{\zeta}_3) \wedge \tau + \sum_i \phi_i^C \wedge \psi_i^J.$$

Here ϕ_i^C is a form from C and ψ_i^J is from $J(C)$, and it is either $\phi_i^C = \bar{\zeta}_3$ or else $\phi_i^C = \bar{\zeta}_3 \wedge \zeta_l, l \neq 3$, and therefore it is of volume 0, because of the orthogonality assumption. We have then : $[R(Z(1)), \bar{\zeta}_3 \wedge \tau \wedge \omega_3^J] = [R(C \times Z), b^*(\bar{\zeta}_3 \wedge \tau \wedge \omega_3^J)] - [R(C \times Z), \bar{\zeta}_3 \wedge \tau \wedge \omega_3^J] = [R(C \times Z), b^*(\bar{\zeta}_3 \wedge \tau \wedge \omega_3^J)] = [-R(Z), \tau]$, because $[R(C \times Z), \phi^C \wedge \psi^J] = [R(Z), \psi^J] \int_C \phi^C$. □

Proof of Proposition 1.1. The previous lemma along with equation (1.2) implies that $R(Z(m))$ gives a non-zero element of $H^{\{g,m+1,1\}}(J(C))/H^{\{g-1,m,0\}}(J(C))$ if $\tau \wedge \omega_3^J \wedge \ldots \omega_{m+2}^J$ is orthogonal to $N^{g-m-1}H^{2g-m-2}(J(C))$. The assumptions on the primitive cohomology imply that $N^{g-m-1}H^{2g-m-2}(J(C)) = \Theta^{g-m-1}H^m(J(C))$. Since $\tau \wedge \omega_3^J \wedge \ldots \omega_{m+2}^J \cdot \Theta^{g-m-1} = 0$, it follows that the condition is indeed satisfied. □

The next proposition shows that the hypothesis on the primitive cohomology of Proposition 1.1 holds for the Jacobian of a generic hyperelliptic curve. The reader may refer to [13] for the definition and basic properties of the Hodge group.

Proposition 1.2. *Let C be a generic hyperelliptic curve of genus g. Then the Hodge group of $J(C)$ is isomorphic to $Sp(2g, \mathbb{Q})$, hence $P^i(J(C))$ is an irreducible Hodge structure for $0 \leq i \leq g$.*

Proof. We shall use induction on g, the result for $g = 1$ and 2 being well known. Assume that $g \geq 3$ and the result is known for smaller g. We degenerate C to a stable curve C_o with 3 smooth irreducible components C_1, C_2 and C_3, with C_1 and C_3 of genus 1, and C_2 of genus $g - 2$ intersecting each of C_1 and C_3 transversally in a single Weierstrass point. By choosing a path in a suitable parameter space, we can identify $V := H^1(C, \mathbb{Q})$ as a symplectic vector space with $V_1 \oplus V_2 \oplus V_3$, where $V_i = H^1(C_i, \mathbb{Q})$, $i = 1, 2, 3$. Let D_1 and D_3 be generic hyperelliptic curves of genus $g - 1$, with D_i specializing to $C_i \cup C_2$, $i = 1, 2$. Let $C' = D_1 \cup C_3$ and $C'' = C_1 \cup D_3$, the two components of each curve intersecting transversally in a single point. Again, by choosing paths we may identify $H^1(C', \mathbb{Q})$ and $H^1(C'', \mathbb{Q})$ with $V_1 \oplus V_2 \oplus V_3$ in such a way that $H^1(D_i, \mathbb{Q})$ is identified with $V_i \oplus V_2$.

Now consider the family of Jacobians. Since C is generic, G, the Hodge group of $J(C)$, contains the Hodge groups G' and G'' of $J(C')$ and $J(C'')$ respectively. Using induction and the above identifications, we see that G contains both $Sp(V_1 \oplus V_2)$ and $Sp(V_2 \oplus V_3)$. One easily checks, by an explicit computation using Lie algebras, that the smallest subgroup of $GL(V)$ containing both these two subgroups is $Sp(V)$. The Hodge group is always contained in $Sp(V)$ so G must equal $Sp(V)$.

From the representation theory of symplectic groups it follows that $P^i(J(C))$ is an irreducible representation of the Hodge group, $0 \leq i \leq g$. Since the sub-Hodge structures of H^i are precisely the subrepresentations of the Hodge group, it follows that the $P^i(J(C))$'s are also irreducible as Hodge structures.   □

1.3. On the real regulator image of the basic hyperelliptic cycle.
Let $f : C \to \mathbb{P}^1$ be the double cover associated with a hyperelliptic curve. We fix two ramification points w_1 and w_2 on C and choose a standard parameterization on $\mathbb{P}^1$ so that $f(w_1) = 0$ and $f(w_2) = \infty$. The points w_1 and w_2 are the distinguished Weierstrass points, and $\epsilon := [w_1 - w_2]$ is the associated element of order two in $Pic^0(C)$.

It is convenient to identify $J(C) = Pic^0(C)$ with $Pic^1(C)$ by adding w_1. We embed C in the natural way in $Pic^1(C)$, and for $t \in Pic^1(C)$ we let C_t be the translate of C by $[t - w_1]$. Now consider $W_1 := C = C_{w_1}$ and $W_2 := C_{w_2}$, the ϵ translate of C, and fix a point $t \in C$. Observe that the intersection $C_t \cap W_1 \cap W_2$ is the point w_1. We shall follow the convention to indicate a rational function on C_t by using the same name given to the corresponding function on C.

Consider $K := W_1 \otimes f + W_2 \otimes f$. K is the *basic hyperelliptic cycle* of [4], where it was provedthat it is a non trivial indecomposable element of $CH^g(J(C), 1)_{\mathbb{Q}}$ for generic C. There it was shown that the primitive contribution of the standard regulator image of K does not vanish by studying an infinitesimal invariant of Griffiths type associated with the relevant normal function. The following proposition shows that K is real regulator indecomposable. Here τ_C is of the form considered in section 1.2.

Proposition 1.3. *For C a generic hyperelliptic curve of genus $g \geq 2$ there is τ_C with $[R(K), \tau_C] \neq 0$.*

By definition $[R(K), \tau] = 2 \int_C log|f|\tau_{|C}$; we shall prove that it is not trivial by means of a reduction process to the case of elliptic curves.

Let E_λ be the elliptic curve with affine equation $y^2 = x(x - 1)(x - \lambda)$. Define $f_\lambda := x$ as a rational function on E_λ. The form associated with the Θ divisor is here $\theta_\lambda = (i/2)\omega_\lambda \wedge \bar{\omega}_\lambda$. It is an invariant form on E_λ of volume one. We write $I(\lambda) := \int_{E_\lambda} \log|f_\lambda|\theta_\lambda$.

Lemma 1.4. *$I(\lambda)$ varies with λ.*

Proof. Multiplication of x by λ^{-1} shows that E_λ and $E_{\lambda^{-1}}$ are isomorphic models of the same curve E. On E the volume forms coincide, while $f_\lambda = \lambda f_{\lambda^{-1}}$. Thus

$$I(\lambda) = \int_E \log|\lambda|\theta + I(\lambda^{-1}) = \log|\lambda| + I(\lambda^{-1}),$$

hence $I(\lambda)$ cannot be constant.   □

We define $I(h, \tau) := \int_C \log|h|\tau_C$, for any rational function h on C.

Lemma 1.5. *If C is a generic curve of genus 2, then $I(f, \tau) \neq 0$.*

Proof. We prove it for a bielliptic curve C which is a double cover of $E_1 := E_{\lambda_1}$ and of $E_2 := E_{\lambda_2}$. Consider the diagram

$$
\begin{array}{ccccc}
E_2 & \xleftarrow{\ k_2\ } & C & \xrightarrow{\ k_1\ } & E_1 \\
{\scriptstyle f_2}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f_1}\big\downarrow \\
\mathbb{P}^1 & \xleftarrow{\ h\ } & \mathbb{P}^1 & \xrightarrow{\ h\ } & \mathbb{P}^1
\end{array}
$$

Here f_i is ramified over $\{0, 1, \infty, \lambda_i\}$, h is the double cover ramified over λ_1 and λ_2, and $f : C \to \mathbb{P}^1$ is the hyperelliptic cover ramified at $h^{-1}(\{0, 1, \infty\})$. On the range of h we have already fixed a standard parameter, we choose a standard parameter on the domain of h so that 0 maps to 0, and similarly for 1 and for ∞. In this manner f is a well defined rational function on C, and we denote by $\bar{f}$ its transform under the involution of $\mathbb{P}^1$ associated with h. Letting $g := hf$, we see that $f\bar{f} = cg$, c a constant.

One can take τ_C to be the form $(k_1^*(\theta_1) - k_2^*(\theta_2))$ on $J(C)$ and thus

$$
I(g, \tau_C) = \int_C \log |g| (k_1^*(\theta_1) - k_2^*(\theta_2)) \neq 0 \,.
$$

It then follows that $I(f, \tau_C) \neq 0$ for the general bielliptic curve C because

$$
I(f, \tau_C) + I(\bar{f}, \tau_C) = I(g, \tau_C) + \log|c| \int_C \tau_C = I(g, \tau_C) \,. \quad \square
$$

Proof of Proposition 1.3. The proof for arbitrary genus is obtained by induction. Starting from a hyperelliptic curve $G \to \mathbb{P}^1$ and a double cover $h : \mathbb{P}^1 \to \mathbb{P}^1$ we construct the commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\ \pi\ } & G \\
{\scriptstyle f_C}\big\downarrow & & {\scriptstyle f_G}\big\downarrow \\
\mathbb{P}^1 & \xrightarrow{\ h\ } & \mathbb{P}^1
\end{array}
$$

Here C is the normalization of the Cartesian product, hence π is branched at $4 - 2m$ points, where m is the number of ramification points of h which coincide with points of ramification for f_G. We have $g(C) = 2g(G) + 1 - m$.

By induction, Proposition 1.3 holds for G. It then also holds for C by using the arguments given for Lemma 1.5, where we now take τ_C to be the form $\pi^* \tau_G$, the lift to $J(C)$ of τ_G. $\quad \blacksquare$

1.4. **Indecomposable elements on hyperelliptic Jacobians.** For a g dimensional abelian variety A, the following decomposition of the higher Chow groups is a consequence of the motivic decomposition of the diagonal due to Beauville [1], and Deninger and Murre [5]:

$$
(1.3) \qquad\qquad CH^k(A, m)_{\mathbb{Q}} = \bigoplus_s CH^k(A, m)_s
$$

Here $CH^k(A, m)_s$ is the subspace of $CH^k(A, m)_{\mathbb{Q}}$ on which $[n]^*$ (resp. $[n]_*$) acts by multiplication by n^{2k-s} (resp. $n^{2g-2k+s}$). The Fourier transform of Mukai and Beauville induces isomorphisms:

$$
(1.4) \qquad\qquad \mathcal{F} : CH^k(A, m)_s \xrightarrow{\ \cong\ } CH^{g-k+s}(\hat{A}, m)_s
$$

where $\hat{A}$ is the dual abelian variety.

For Θ a symmetric ample divisor, the motivic hard Lefschetz theorem of Künnemann [11] implies that intersecting with powers of Θ gives isomorphisms:

$$(1.5) \qquad \cdot\,\Theta^{g+s-2k} : CH^k(\mathcal{A}, m)_s \xrightarrow{\cong} CH^{g+s-k}(\mathcal{A}, m)_s \,,\;\; 0 \le 2k - s \le g \,.$$

It follows from the definitions that the decomposition (1.3) and the isomorphisms (1.4) and (1.5) preserve decomposable cycles as defined in (1.1), hence

$$(1.6) \qquad\qquad CH^k_{\mathrm{ind}}(A, 1)_{\mathbb{Q}} = \bigoplus_s CH^k_{\mathrm{ind}}(A, 1)_s$$

$$(1.7) \qquad\qquad \mathcal{F} : CH^k_{\mathrm{ind}}(A, 1)_s \xrightarrow{\cong} CH^{g-k+s}_{\mathrm{ind}}(A, 1)_s$$

$$(1.8) \qquad \cdot\,\Theta^{g+s-2k} : CH^k_{\mathrm{ind}}(\mathcal{A}, 1)_s \xrightarrow{\cong} CH^{g+s-k}_{\mathrm{ind}}(\mathcal{A}, 1)_s \,,\;\; 0 \le 2k - s \le g \,.$$

Proposition 1.4. *Let A be a g-dimensional abelian variety. Then $CH^g_{\mathrm{ind}}(A, 1)_s = 0$ for $s < 2$ or $s > g$.*

Proof. We use equation (1.7) i.e. $\mathcal{F}(CH^g_{\mathrm{ind}}(A, 1)_s) = CH^s_{\mathrm{ind}}(\hat{A}, 1)_s$. If $s < 1$, $CH^s(\hat{A}, 1)$ is itself zero. The action of $[n]^*$ on $CH^1(\hat{A}, 1) = \mathbf{C}^*$ is trivial hence $CH^1(\hat{A}, 1)_1$ is also zero. We conclude the proof by observing that for any g-dimensional variety X, $CH^s(X, 1) = 0$ for $s > g + 1$ and $CH^{g+1}_{\mathrm{ind}}(X, 1) = 0$ $\qquad\square$

Remark. A conjecture of C. Voisin [17] says that $CH^2_{\mathrm{ind}}(X, 1)$ should be countable for any smooth projective variety X. For an abelian variety A, the injectivity of the rational regulator on $CH^k(A, 1)_2$ would imply that $CH^k_{\mathrm{ind}}(A, 1)_2$ is countable. If $g = 2$, the proposition shows that then $CH^2_{\mathrm{ind}}(A, 1)_{\mathbb{Q}}$ would also be countable.

For the rest of this section, C will be a hyperelliptic curve of genus g and K the basic hyperelliptic cycle in $CH^g(J(C), 1)$.

Lemma 1.6. *The component of K in $CH^g_{\mathrm{ind}}(J(C), 1)_s$ is zero for all $s \ne 2$. Consequently, for any integer n, $[n]_*(K) = n^2 K$ in $CH^g_{\mathrm{ind}}(J(C), 1)_{\mathbb{Q}}$.*

Proof. Consider the following copies of C embedded in $C \times C$: $X_1 = C \times \{w_1\}$, $X_2 = \{w_1\} \times C$, $X_3 = C \times \{w_2\}$, $X_4 = \{w_2\} \times C$, $X_5 = \Delta$, and $X_6 = \Delta'$. Here w_1 and w_2 are two distinct Weierstrass points, Δ is the diagonal, and Δ' is the image of C via the embedding $x \mapsto (x, \sigma(x))$, with σ the hyperelliptic involution. If f is a Weierstrass function with $div(f) = 2w_1 - 2w_2$ then one easily checks that $Z = \sum_{i=1}^6 X_i \otimes f$ is an element of $CH^2(C \times C, 1)$. If we use w_1 to embed C in $J(C)$, then the image of Z in $CH^g(J(C), 1)$ is equal to $2K - (1/2)[2]_*(K)$.

The involution $\tau = (id, \sigma)$ of $C \times C$ preserves Z, hence Z must be the pullback of a cycle from $C \times C / \tau \cong C \times \mathbb{P}^1$. $CH^2(C \times \mathbb{P}^1, 1)$ is always decomposable for any curve C, hence Z must also be decomposable. Thus $4K = [2]_*(K)$ in $CH^g_{\mathrm{ind}}(J(C), 1)_{\mathbb{Q}}$. This implies that all the components of K except the one in $CH^g_{\mathrm{ind}}(J(C), 1)_2$ must be zero. $\qquad\square$

For $T = (t_1, t_2, \ldots, t_m)$, $P = (p_1, p_2, \ldots, p_m)$ points on C^m, let $K_P(T) = K * ([t_1 - p_1] - e) * ([t_2 - p_2] - e) * \ldots * ([t_m - p_m] - e) \in CH^g(J(C), 1)$, where e is the origin in $J(C)$. The following theorem is the main result of this section.

Theorem 1.2. *If C is a generic hyperelliptic curve and T a generic point of C^m, $1 \le m \le g - 2$, then $K_P(T)$ is indecomposable and the set of all such cycles with T*

varying and P fixed forms an uncountable subset of $CH^g_{\text{ind}}(J(C), 1)_{\mathbb{Q}}$. Furthermore, $CH^k_{\text{ind}}(J(C), 1)_s$ is uncountable for $3 \leq s \leq k \leq g$.

Proof. The first part of the theorem follows directly by combining Propositions 1.1, 1.2 and 1.3.

Let $I = \oplus^g_{s=1} CH^g(J(C))_s$ be the zero cycles of degree 0 (with $\mathbb{Q}$ coefficients). Then for $m \geq 1$, $I^{*m} = \oplus^g_{s=m} CH^g(J(C))_s$ and if $T \in C^m$, then $K_P(T) \in CH^g(J(C), 1)_{\mathbb{Q}} * I^{*m}$. Using Proposition 1.4, Lemma 1.6 and the fact that the subscripts are additive under Pontryagin products, we see that if $m = g - 2$ then the only nonzero component of $K_P(T)$ in $CH^g_{\text{ind}}(J(C), 1)_{\mathbb{Q}}$ is the one in $CH^g_{\text{ind}}(J(C), 1)_g$. The first part of the theorem then implies that the image of $K * CH^g(J(C))_{g-2}$ in $CH^g_{\text{ind}}(J(C), 1)_g$ is uncountable. We then use that $CH^g(J(C))_s = CH^g(J(C))^{*s}_1$, $s \geq 1$, and descending induction on s starting from $s = g$ to show that the image of $K * CH^g(J(C))_{s-2}$ in $CH^g_{\text{ind}}(J(C), 1)_s$ is uncountable for $3 \leq s \leq g$.

The statement for $k < g$ follows by using the Fourier transform and the motivic hard Lefschetz theorem: Letting $k = s$, we see by (1.7) that $CH^s_{\text{ind}}(J(C), 1)_s$ is uncountable for $3 \leq s \leq g$. By (1.8), intersection with Θ^{g-s} induces an isomorphism from $CH^s_{\text{ind}}(J(C), 1)_s$ to $CH^g_{\text{ind}}(J(C), 1)_s$, hence intersection with Θ^{k-s} must induce an injection from $CH^s_{\text{ind}}(J(C), 1)_s$ to $CH^k_{\text{ind}}(J(C), 1)_s$, $3 \leq s \leq k \leq g$. Since $CH^s_{\text{ind}}(J(C), 1)_s$ is uncountable, it follows that $CH^k_{\text{ind}}(J(C), 1)_s$ is also uncountable. $\square$

Remark. Proposition 1.4 shows that the theorem is optimal for $k = g$. It should also be optimal for $k < g$ — the proof for $s > k$ still works but we do not know how to handle the $s < 2$ case.

2. DECOMPOSABILITY SPECIALIZES

Our aim is to show that decomposability specializes. Consider a flat and projective family $\mathcal{X} \to A$, where A is a smooth curve, $\mathcal{X}$ is non singular and so is X_0, the fibre over p_0.

Theorem 2.1. *If the restriction of an element $\mathcal{Q} \in CH^d(\mathcal{X}, 1)_{\mathbb{Q}}$ to the generic fibre is decomposable then restriction of $\mathcal{Q}$ to the central fibre is also decomposable.*

2.1. One can replace $\mathcal{Q}$ by a multiple and A by an open subset of a finite cover of the original A so that now the following holds:

Assumption. Over $U := A - \{p_0\}$ the restriction $\mathcal{Q}_U$ is equivalent in $CH^d(\mathcal{X}_U, 1)$ to an element $\mathcal{W} := \sum \mathcal{Z}_j \otimes f_j$, where $\mathcal{Z}_j$ are irreducible subvarieties which intersect properly X_0 and f_j are rational functions lifted from A and regular on U.

We meet now the problem that $\mathcal{W}$ may have a boundary B on $\mathcal{X}$, and then B is supported on X_0. On the other hand $\mathcal{Q}$ is a cycle, and so it has no boundary. Since on $\mathcal{X}_U$ the classes $\mathcal{Q}$ and $\mathcal{W}$ coincide by hypothesis, then under the boundary map $CH^d(\mathcal{X}_U, 1) \to CH^{d-1}(X_0)$ the image of $\mathcal{W}$ is trivial, that is B is rationally equivalent to 0 on the central fibre. Let R be a relation of rational equivalence on X_0 which kills B, then $\mathcal{W} - R$ has no boundary and thus $\mathcal{W} - R - \mathcal{Q}$ is a cycle for $CH^d(\mathcal{X}, 1)$ whose class is represented by a cycle M coming from $CH^{d-1}(X_0, 1)$. We will show that M and R can be moved conveniently for our purposes, but to do this we need to embed $\mathcal{X}$ in a larger space $\mathcal{Y} \to A$. Our first result is that

we can work on $\mathcal{Y}$ and with a *cycle* $\mathcal{W}^Y := \sum \mathcal{Z}_j \otimes f_j$ as described above. The improvement is that $\mathcal{W}^Y$ has no boundary now and that it is equivalent to $\mathcal{Q}$ on $\mathcal{Y}$. The second issue is to compute the restriction of $\mathcal{W}^Y$ to X_0 so to check that it is indeed of decomposable type. The difficulty that we meet is the fact that each f_j may have a boundary on the central fibre, we overcome this trouble by means of some test curves.

2.2. We find useful to work for a while with $G(T)$ the K-theory of coherent sheaves on a quasiprojective scheme T and to use the topological filtration $F_m G_1(T)$, this is the subgroup generated by the images of $i_{Z,*} : G_1(Z) \to G_1(T)$, where $i_Z : Z \to T$ are closed subschemes of T of dimension at most m. The relations with Bloch's $CH^p(X,1)$ are now recalled.

We know from Soulé [16] that the Quillen coniveau spectral sequence

$$E_1^{p,q} = \coprod_{x \in X_p} K_{-p-q} k(x) \Rightarrow G_{-p-q}(X)$$

degenerates modulo torsion for $p + q \leq 2$, in particular we need

$$G_1(X)_{\mathbb{Q}} = \bigoplus_{p=0}^{d} E_2^{p,-p-1} \otimes \mathbb{Q}$$

Moreover for $m = 0, 1, 2$ the coniveau filtration of the spectral sequence coincides rationally with the γ filtration. Using Bloch's isomorphism

$$\bigoplus_i gr_\gamma^i G_m(X)_{\mathbb{Q}} \simeq G_m(X)_{\mathbb{Q}} \simeq \bigoplus_i CH^i(X,m)_{\mathbb{Q}}$$

one has then

$$E_2^{p,-p-1} \otimes \mathbb{Q} \simeq CH^p(X,1)_{\mathbb{Q}}$$

Let $i_D : D \to T$ be the inclusion of an effective Cartier divisor, then $i_D^* G_m(T) \to G_m(D)$ is defined. Next proposition, see 2-11 in [15], implies that if each component of a subscheme Z of dimension m intersects properly D, then $i_D^*(i_{Z*}(G_1(Z))$ lands in $F_{m-1} G_1(D)$.

Proposition 2.1 (Quillen)**.** *Consider a cartesian diagram of quasiprojective schemes*

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
f' \downarrow & & f \downarrow \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

Assume that f is proper, that g is of finite tor-dimension and that Y' and X are tor-independent over Y (i.e. $Tor_i^{O_{Y,y}}(O_{(Y',y')}, O_{(X,x)}) = 0$ for $i \geq 1$)
    Then

$$g^* f_* = f'_* g'^* : G_m(X) \to G_m(Y')$$

2.3. **A larger space.** We come back to consider $\mathcal{X} \to A$, by gluing it transversally with $X \times A$ along the fibre over p_0 we construct $\mathcal{Y}$, so that $g : \mathcal{Y} \to A$ has central fibre the scheme Y, whose reduced support is X_0.

The diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \mathcal{Y} := \mathcal{X} \cup_{X \times \{p_0\}} (X \times A)
\end{array}
$$

satisfies the hypotheses of Proposition 2.1. Indeed one has only to check vanishing of Tor_1, because Y is a Cartier divisor in $\mathcal{Y}$. Let t be a local parameter at p_0, then t generates the ideal of Y, and we see that $Tor_1 = 0$ because t is not a 0-divisor in the local rings of $\mathcal{X}$.

Applying devissage we find then:

$$
\begin{array}{ccc}
G_q(X) & \longleftarrow & G_q(\mathcal{X}) \\
\simeq \downarrow & & \downarrow \\
G_q(Y) & \longleftarrow & G_q(\mathcal{Y})
\end{array}
$$

We use next the setting above to check that the specialization map $G_1(\mathcal{X}) \to G_1(X)$ is compatible with the topological filtration F_m. The useful fact is that on $\mathcal{Y}$ it is easy to move to general position with respect to X. This way of moving turns out to be also the trick that we need to show that decomposability specializes.

We work by induction on m and consider z in the graded quotient $F_{m+1}/F_m G_1(\mathcal{X})$, then z is represented by a Quillen cycle $\sum Z_i \otimes f_i$, with $\sum div(f_i) = 0$ where Z_i are irreducible subvarieties of dimension $m + 1$. Now:

Lemma 2.1. *On $X \times A$ a Quillen chain $Z \otimes f$ is equivalent to a chain whose support intersects properly $X \times \{p_0\}$.*

Proof. The only problem is when $Z \subset X \times \{p_0\}$. Consider the symbol $\{g, f\}$ on $Z \times A$, where g is a rational function with simple 0 at p_0 and therefore $div(g) = p_0 + \sum m_i q_i$. The boundary of $(Z \times A) \otimes \{g, f\}$ is $(Z \times \{p_0\}) \otimes f + \sum (Z \times \{q_i\}) \otimes f^{m_i} - (div(f) \times A) \otimes g$, and thus $(Z \times \{p_0\}) \otimes f$ is equivalent to a chain of the stated kind. □

In this way the image of z in $F_{m+1}/F_m G_1(\mathcal{Y})$ is supported on a scheme T of dimension $m + 1$ with the property that each component of T meets properly the central divisor X, therefore the restriction of z belongs to $F_m G_1(X)$ because it is supported on the intersection $T \cap X$.

The same proof yields that the chain $M + R$ which was discussed above is equivalent on $\mathcal{Y}$ to a sum of Quillen chains of the preceding type $(div(f) \times A) \otimes g$. Notice that here we may have had to replace A by a smaller open set, with this proviso we have then proved:

Proposition 2.2. *The restriction of Q to X coincides with the restriction from $\mathcal{Y}$ to Y of a cycle $\mathcal{W}^Y := \sum \mathcal{Z}_j \otimes f_j$, where $\mathcal{Z}_j$ are irreducible varieties in $\mathcal{Y}$ and where f_j is pull back of a rational function of the same name from A under the flat projection $\mathcal{Z}_j \to A$. The functions f_j are regular away from p_0.*

2.4. **Computing the restriction of $\mathcal{W}^Y$.** Let $(\cup_j Z_j) \cap X_0 = \cup_l H_l$ be the decomposition in irreducible components, then we know from the preceding discussion that the restriction morphism $CH^d(\mathcal{Y}, 1) \to CH^d(X, 1)$ sends $\mathcal{W}^Y$ to a cycle supported on $\cup_l H_l$, and therefore to a cycle $\sum_l H_l \otimes h_l$. Next proposition completes the proof of the theorem.

Proposition 2.3. *The rational functions h_l are constant functions.*

Proof. By taking general linear sections of $\mathcal{Z} = \cup \mathcal{Z}_j$ one may construct curves $B = \cup B_j$ such that $B \to A$ is finite. Our program is to prove that at the points of intersection of B with H_l the value of the relevant function h_l is constant, independent of the chosen section B.

We may assume that B is smooth outside the inverse image of p_0. Let Q be the element of $G_1(B)$ determined by $\mathcal{W}^Y$, and thus Q is such that its restriction to the smooth part is given by functions f_j on the components B_j of B, each f_j being the pull-back of a rational function with the same name on A. Using devissage we restrict Q to $G_1(p) \simeq \mathbb{C}^*$, where p is a point in the inverse image of p_0. Then the claim is that for any such point p this restriction is equal to $(\prod f_j{}^{n_j})(p_0)$. Here the n_j's may depend on p and they are such that the product doesn't have a pole or zero at p_0. Note that contravariant functoriality for local complete intersections yields here the equality $(\prod f_j{}^{n_j})(p_0) = \prod(h_l(p))$, where we set $h_l(p) = 1$ if $p \notin H_l$. This argument shows that at the points of intersection of curves like B with H_l the values of the functions h_l lie in a countable set, and therefore h_l must be constant on the irreducible component H_l.

To prove the claim, we let C be the normalization of an irreducible component of the fibre product of all the B_i's over A. C is flat over A, so we can replace C by A and B by the fibre product of A and B over C. The advantage is that here we are reduced to the case that there is only one point in the inverse image of (the new) 'p' because we now have sections, and so the claim follows by functoriality from the simple case, which we describe next.

The simple case is when each B_i is an isomorphic copy of A and moreover the scheme $B := \cup B_i$, is constructed by gluing at one point q ($= p_0$ on A). Let the projection map be $\pi : B \to A$, then we write $U := A - \{p_0\}$, and $V_i = B_i - \{q\}$, and set $V := \cup V_i = B - \{q\}$. The map $\pi : B \to A$ is the identity on each component B_i. Give rational functions f_i on B_i, (A is not necessarily complete, and we assume that the only zero or pole of f_i is at q) and assume that $\sum div(f_i) = 0$. Then $\{f_i\}$ defines an element f say in $G_1(V)$, which comes from $G_1(B)$, because it has boundary 0 in the exact sequence $G_1(B) \to G_1(V) \to G_0(q)$. Consider now p_0 as a Cartier divisor on A and let q^* be the scheme, which is the pull back of p_0 on B. The map $G_1(B) \to G_1(q*)$ is well defined because q^* is a local complete intersection in B. By devissage $G_1(q^*) = G_1(q) = \mathbb{C}^*$; the question is to understand the value of the image of f in $G_1(q) = \mathbb{C}^*$. Each f_i is a function f_{iA} on A. The answer is consider the product $f_A := \Pi f_{iA}$, then f_A is a rational function which is in fact regular at p_0, because of our assumption, and then we have that the image value is $f_A(p_0)$. The reason is once more the commutativity from Proposition 2.1 (it is so below at p_0 and then it must have been so at q). $\qquad\square$

3. The 4-configuration

For C a generic hyperelliptic curve of genus $g \geq 3$, we have constructed indecomposable elements in $CH^g(J(C), 1)_{\mathbb{Q}}$ which are in the kernel of the regulator map.

We shall now construct such elements in $CH^g(J(C), 1)_{\mathbb{Q}}$, for more general curves, in particular for generic curves of genus 3 and 4. The natural parameter space for our construction will be a certain Hurwitz scheme of degree 4 covers of $\mathbb{P}^1$. We shall show that one can specialize to a hyperelliptic curve in such a way that our cycle specializes to $K - K_t$, allowing us to use the results of the previous sections to prove indecomposability.

3.1. Construction of the 4-configuration.
Let C be a smooth projective curve of genus $g \geq 3$ and let $D = (a_1 + a_2) - (b_1 + b_2)$ be a divisor on C such that $[D] = \epsilon$ is a point of order 2 in $Pic(C)$. Let f be a rational function on C such that $div(f) = 2D$. To this data we shall associate a natural element of $CH^g(Pic^3(C), 1)$ supported on four copies of C.

Let $i(y, z) : C \to Pic^3(C)$ be the map $i(y, z)(x) = x + y + z$, $C(y, z) := i(y, z)(C)$, and let $j : C \to Pic^3(C)$ be the map $j(x) = -x + 2(a_1 + a_2)$, $G := j(C)$. We consider $C(a_1, a_2)$, $C(b_1, b_2)$ and G. Translation by ϵ maps $C(a_1, a_2)$ to $C(b_1, b_1)$, and we let G_ϵ be the image of G. We shall use the convention that f represents the rational function on each of the preceding curves which maps to f under the chosen isomorphism with C. Thus, we set $Z_1 := C(a_1, a_2) \otimes f$, $Z_2 := G \otimes f$, $Z_3 := C(b_1, b_2) \otimes f$, $Z_4 := G_\epsilon \otimes f$.

Proposition 3.1. $Z_D := \sum_{i=1}^{4} (-1)^i Z_i$ is a higher cycle.

Note that Z_D also depends on the choice of the rational function f, but we can neglect this since multiplying f by an element of $\mathbb{C}^*$ amounts to the addition of a decomposable element to Z_D.

Proof. The only possible difficulty is to see where the curves intersect. $C(a_1, a_2)$ intersects G in two points; on both curves the points come from a_1 and a_2 under the isomorphism with C, but the point which comes from a_1 in G comes from a_2 in $C(a_1, a_2)$ and conversely. A similar statement holds for $C(b_1, b_2) \cap G$, and then intersections with G_ϵ can be recovered by using ϵ-symmetry. Note that if C is not hyperelliptic then $C(a_1, a_2) \cap C(b_1, b_2) = \emptyset$ and $G \cap G_\epsilon = \emptyset$. $\square$

Remark. By translation of Z_D by a a zero cycle ξ of degree -3 on C, we obtain an element $Z_{D,\xi}$ of $CH^g(J(C), 1)$. The component of $Z_{D,\xi}$ in $CH^g(J(C), 1)_2$ is always zero, hence it is in the kernel of the (rational) regulator map. To see this, first note that this component is preserved by translation since $CH^g(J(C), 1)_s = 0$ for $s < 2$ (c.f. proof of Proposition 1.4), hence is independent of ξ. One easily checks that $[-1]^*(Z_{D,-3a_1}) = -Z_{D,-3a_2}$, This implies the desired fact, since the action of $[-1]^*$ on $CH^g(J(C), 1)_2$ is trivial.

3.2. A Hurwitz scheme.
In order to construct the specialization needed to prove indecomposability, we shall use certain Hurwitz schemes parametrizing the ramified coverings of $\mathbb{P}^1$ corresponding to the functions f as above. We refer the reader to [7, Section 1] for the basic facts about Hurwitz schemes.

Let $g \geq 2$ be an even integer and H_g be the Hurwitz scheme whose points correspond to degree 4 covers of $\mathbb{P}^1$ branched over $n = 2g + 4$ distinct points, such that the inverse image of $n - 2$ of these points consists of three points and the inverse image of each of the remaining points consists of two points, each of ramification degree 2. From the Riemann-Hurwitz formula it follows that such a cover $C \to \mathbb{P}^1$ is of genus g.

Proposition 3.2. *For each $g \geq 2$, H_g consists of two components.*

Proof. Let $n = 2g+4$. Recall that we have a finite etale map $\delta : H_g \to \mathbb{P}^n - \Delta$, where $\mathbb{P}^n$ is thought of as $(\mathbb{P}^1)^{(n)}$ and Δ is the discriminant locus. Let $P = \{p_1, p_2, \ldots, p_n\}$ be an element of $\mathbb{P}^n - \Delta$ and let $x \in \mathbb{P}^1$ be distinct from the p_i's. We may choose loops σ_i based at x and going around p_i in such a way so that $\sigma_1, \sigma_2, \ldots \sigma_n$ generate $G = \pi_1(\mathbb{P}^1 \backslash \{p_1, p_2, \ldots, p_n\}, x)$ with the single relation as $\sigma_1 \sigma_2 \cdots \sigma_n = 1$. Now degree 4 covers of $\mathbb{P}^1$ branched over P correspond to transitive representations of G in the symmetric group Σ_4. For the covers to correspond to points of H_g, the images of $n - 2$ of the σ_i's must be transpositions and the images of the other two must be products of 2 disjoint transpositions. Two such representations give isomorphic covers if they differ by an inner automorphism of Σ_4. Thus $\delta^{-1}(P)$ can be identified with classes of n-tuples $(s_1, s_2, \ldots, s_n)$ of elements of Σ_4, all but two of the s_i's being transpositions, the remaining two being products of two disjoint transpositions, and $s_1 s_2 \cdots s_n = 1$. Two such n-tuples are identified if they differ by coordinatewise conjugation by an element of Σ_4.

The action of the monodromy on $\delta^{-1}(P)$ contains elements Γ_i, $1 \leq i \leq n - 1$, which act on n-tuples as above by:

$$\Gamma_i(s_1, s_2, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_i s_{i+1} s_i^{-1}, s_i, \ldots, s_n).$$

To show that H_g has two components we use the Γ_i's to prove that the monodromy action on $\delta^{-1}(P)$ has two orbits. The proof is a case by case analysis; we shall describe the main steps and leave some simple verifications to the reader. Let $t_{i,j}$, $i \neq j$, denote the transposition which switches i and j and let v_1, v_2 and v_3 be the permutations $(1\,2)(3\,4)$, $(1\,3)(2\,4)$ and $(1\,4)(2\,3)$ respectively. Let $V = \{v_1, v_2, v_3\}$. Using the action of the Γ_i's one sees that each orbit contains an n-tuple S such that s_{n-1} and s_n are in V. Upto conjugation, we may assume that $s_n = v_1$ and $s_{n-1} = v_1$ or v_2. Suppose $s_{n-1} = v_2$, hence $s_1 s_2 \cdots s_{n-2} = v_3$. Let K be the subgroup of Σ_4 generated by $s_1, s_2, \ldots s_{n-2}$. Using the fact that $v_3 \in K$ and K is generated by transpositions we analyze the two possibilities for the action of K on $\{1, 2, 3, 4\}$ (i) transitive and (ii) intransitive. In both cases ((i) requires some computations with the Γ_i's) we conclude that we may assume that at least one of $t_{1,4}$ or $t_{2,3}$ occurs among the s_i's (without changing s_{n-1} and s_n). Again using the action, we may assume that $i = n - 2$. Then replacing S, by $\Gamma_{n-2}(\Gamma_{n-1}(S))$ we may also assume that $s_{n-1} = v_1$.

As above, let K be the subgroup of Σ_4 generated by $s_1, s_2, \ldots s_{n-2}$ and note that now $s_1 s_2 \cdots s_{n-2} = 1$. We now consider four possibilities for the action of K on $\{1, 2, 3, 4\}$.

(i) The action is transitive.
(ii) The action has a unique fixed point.
(iii) The action has two fixed points.
(iv) The action has no fixed points but fixes two disjoint subsets of two elements each.

(i) Since $s_1 s_2 \cdots s_{n-2} = 1$, it follows by a result of Clebsch (see [7, p.547]) that we may assume

$$(s_1, s_2, \ldots s_{n-2}) = (t_{1,2}, t_{1,2}, t_{1,3}, t_{1,3}, t_{1,4}, \ldots t_{1,4}).$$

Note that a priori we may also have to use an inner automorphism of Σ_4 in order to achieve this, but it is easy to check using the action of the Γ_i's that we may choose

an automorphism preserving v_1. Hence we may assume that

$$S = (t_{1,2}, t_{1,2}, t_{1,3}, t_{1,3}, t_{1,4}, \ldots t_{1,4}, v_1, v_1) .$$

(ii) Without loss of generality, we may assume that the fixed point is 4. Again, by Clebsch' result we may assume that

$$S = (t_{1,2}, t_{1,2}, t_{1,3}, \ldots t_{1,3}, v_1, v_1) .$$

Moreover, it is easy to check that if we replace v_1 by v_2 or v_3, we stay in the same orbit, hence all elements of type (ii) lie in one orbit. Further, note that using the Γ_i's , we can switch pairs of adjacent transpositions, i.e.

$$(t_{1,2}, t_{1,2}, t_{1,3}, t_{1,3}) \sim (t_{1,3}, t_{1,3}, t_{1,2}, t_{1,2}) ,$$

so we may replace S by $(t_{1,3}, \ldots, t_{1,3}, t_{1,2}, t_{1,2}, t_{1,3}, t_{1,3}, v_1, v_1)$. One checks that

$$\Gamma_{n-2}(\Gamma_{n-3}(\Gamma_{n-4}(\Gamma_{n-5}(\Gamma_{n-4}(\Gamma_{n-5}(\Gamma_{n-4}(\Gamma_{n-3}(\Gamma_{n-2}(\Gamma_{n-4}(S)))))))))) $$
$$= (t_{1,3}, \ldots, t_{1,3}, t_{2,4}, t_{1,2}, t_{1,2}, t_{2,4}, v_1, v_1) .$$

Since we have assumed that $n \geq 8$, it follows that the K corresponding to this element acts transitively on $\{1, 2, 3, 4\}$. Hence elements of types (i) and (ii) lie in the same orbit.

(iii) In this case we must have $S = (t, t, \ldots, t, v_1, v_1)$ where t is one of $t_{1,3}$, $t_{2,3}$, $t_{1,4}$, $t_{2,4}$, since the representation is assumed to be transitive. It is clear that all choices are conjugate by an element of Σ_4 which fixes v_1, hence all elements of type (iii) are in the same orbit.

Assume, without loss of generality that $t = t_{1,3}$. Then

$$\Gamma_{n-2}(\Gamma_{n-3}(\Gamma_{n-3}(\Gamma_{n-2}(S)))) = (t_{1,3}, t_{1,3}, \ldots, t_{1,3}, t_{2,4}, t_{2,4}, v_1, v_1) .$$

Since $t_{1,3}$ and $t_{2,4}$ commute, Γ_i, for $1 \leq i \leq n - 2$, just switches s_i and s_{i+1}. Thus we may repeat the above procedure and deduce that S is in the same orbit as elements of the form $(s_1, s_2, \ldots, s_{n-2}, v_1, v_2)$ where each s_i for $1 \leq i \leq n - 2$, is either $t_{1,3}$ or $t_{2,4}$, there being an even number of both kinds.

(iv) Elements of type (iv) are precisely those considered in the previous paragraph, so all such elements lie in the same orbit as elements of type (iii).

We thus see that H_g has at most two components. Now observe that covers of type (iii) have an automorphism of order 2 commuting with the covering map, corresponding to the representation of G in Σ_2 induced from the original representation in Σ_4 by identifying 1 with 2 and 3 with 4. Covers of type (i) have no automorphisms commuting with the covering map, hence H_g has precisely two components.   □

3.3. Indecomposability of the generic 4-configuration.
Let H'_g be the component of H_g corresponding to covers without automorphisms. Then there exists a universal family of curves $\psi : \mathcal{C}_g \to H'_g$ and a corresponding universal degree 4 map $\pi : \mathcal{C}_g \to H'_g \times \mathbb{P}^1$.

We continue using the same notation as in the proof of Proposition 3.2, so $P = \{p_1, p_2, \ldots, p_n\}$, $p_i \in \mathbb{P}^1$, and let p be any other point of $\mathbb{P}^1$. Let X be the curve in $\mathbb{P}^n = (\mathbb{P})^{(n)}$ with points $\{(1 - t)p_1 + tp, (1 - t)p_2 + tp, p_3, p_4, \ldots p_n\}$, $t \in \mathbb{C}$. By the previous proposition, there exists an element of $\delta^{-1}(P)$ such that the monodromy representation of the corresponding cover is given by the n-tuple $S = (t_{2,3}, t_{2,3}, t_{1,2}, t_{1,2}, \ldots, t_{1,2}, v_1, v_1)$ of elements of Σ_4. Let Y' be the component of $\delta^{-1}(X)$ containing this point and let Y be the normalization of X in the function

field of Y'. Then $\mathcal{C} = \psi^{-1}(Y')$ maps to $X \times \mathbb{P}^1$ by the map π and we let $\overline{\mathcal{C}}$ be the normalization of $X \times \mathbb{P}^1$ in the function field of $\mathcal{C}$. Clearly $\psi|_{\mathcal{C}}$ induces a flat and projective morphism $\overline{\psi} : \overline{\mathcal{C}} \to Y$. Let y_0 be a point of Y lying above the point $\{p, p, p_3, p_4, \ldots p_n\}$ of X.

Lemma 3.1. *The fibre of the map $\overline{\psi} : \overline{\mathcal{C}} \to Y$ over y_0 has two smooth components, C_1 and C_2, meeting transversally in a single point lying over p. The map from C_1 to $\mathbb{P}^1$ has degree 2 and is branched over p_3, p_4, $\ldots$, p_n, while the map from C_2 to $\mathbb{P}^1$ is branched over p_{n-1} and p_n. In particular, C_1 is hyperelliptic of genus $g = (n-4)/2$ while C_2 is of genus 0.*

Proof. Let B be small topological disc in Y containing y_0 and such that the only point where the map from B to X is ramified is y_0. By construction of X, if B is small enough then the ramification locus of the map π intersected with $(B - \{y_0\}) \times \mathbb{P}^1$ consists of n disjoint punctured discs, each mapping isomorphically to $B - \{y_0\}$. The closures of all these discs remain disjoint in $B \times \mathbb{P}^1$ except for two of them which meet at (y_0, a). By construction, the product of the local monodromies around these two discs is trivial, hence the map $\pi|_{\overline{\psi}^{-1}(B)} : \overline{\psi}^{-1}(B) \to B \times \mathbb{P}^1$ is unramified over the complement of the closures of all the above discs.

To complete the proof we examine the induced monodromy representation of $\pi_1(\mathbb{P}^1 \backslash \{a, p_3, p_4, \ldots, p_n\}, x)$ in Σ_4. The local monodromy around a is equal to the product of the local monodromies around p_1 and p_2 of the original representation used in the construction of Y and the local monodromies around the other points are the same as those in the original representation. Then we see that the representation is no longer transitive but breaks up into two representations, each of degree 2. The conclusions of the lemma follow immediately on inspection of these two representations. $\qquad\square$

For any point h of H'_g, the covering map $C = (\mathcal{C}_g)_h \to \mathbb{P}^1$ gives us points a_1, a_2, b_1, b_2 on C such that $2[a_1 + a_2] = 2[b_1 + b_2]$. Choosing a function f on C such that $div(f) = 2(a_1 + a_2) - 2(b_1 + b_2)$ allows us to define a 4-configuration Z in $Pic^3(C)$. Note that Z is well defined upto sign in $CH^g_{\mathrm{ind}}(Pic^3(C), 1)$.

Theorem 3.1. *The 4-configuration corresponding to a generic point of H'_g is indecomposable for all $g \geq 3$. Moreover $CH^k_{\mathrm{ind}}(J(C), 1)_s$ is uncountable for $4 \leq s \leq k \leq g$.*

Proof. Let $\overline{\mathcal{C}} \to Y$ be the family constructed in the discussion preceding Lemma 3.1. We blow down the genus 0 curve C_2 in the fibre over y_0 and call the resulting family of curves $\mathcal{C}'$. By replacing Y by a Zariski open subset we may assume that all fibres are smooth and then by replacing Y by a finite cover we may also assume that branch locus of the map $\overline{\mathcal{C}} \to Y \times \mathbb{P}^1$ is a union of sections. Finally, by replacing Y by a further open subset (containing y_0) we may assume that there exist a rational function F on $\mathcal{C}'$ such that $div(F) = 2(A_1 + A_2) - 2(B_1 + B_2) = 2\mathcal{D}$, where $\mathcal{D}$ restricts to the divisor used to define the 4-configuration on each fibre.

F allows us to construct an element $\mathcal{Z}$ of $CH^g(Pic^3(\mathcal{C}'/Y), 1)$ which restricts to the 4-configuration on each fibre. By Lemma 3.1, we see that upto labelling the restrictions of A_1, B_1, A_2, B_2 to $C_1 = \mathcal{C}'_{y_0}$ must be w_1, w_2, t, t, where w_1 and w_2 are distinct Weierstrass points on C_1 and t is a point lying over the point $a \in \mathbb{P}^1$. It is then clear that $f_{y_0} = F|_{C_1}$ has divisor $2w_1 - 2w_1$ and is therefore a Weierstrass function. One easily checks that the element of $CH^g(J(C_1), 1)$ obtained

by translating $\mathcal{Z}_{y_0}$ by $-3w_1$ is equal to $K - K_t$, where K is the basic hyperelliptic cycle and the subscript denotes translation. Since $p_1, p_2, \ldots, p_n$ and p are arbitrary points in $\mathbb{P}^1$, it follows that we may assume that C_1 is a generic hyperelliptic curve of genus g and t a generic point on C_1. By Theorem 1.2 it follows that $\mathcal{Z}_{y_0}$ is indecomposable and then by Theorem 2.1 it follows that $\mathcal{Z}$ restricted to a generic fibre is also indecomposable.

The second statement follows by considering the Pontryagin product of the 4-configuration with the zero cycles $(t_1 - t_0) * (t_2 - t_0) * \cdots * (t_r - t_0)$, $0 \le r \le g - 3$, where $(t_0, t_1, \ldots, t_r)$ is a generic point of C^{r+1}. Taking $r = g - 3 > 0$, we see from the proof of Theorem 1.2 that the components of elements of $Z * I^{*(g-3)}$ give uncountably many elements of $CH^g_{\mathrm{ind}}(J(C), 1)_g$ (note that specialization preserves the decompositions). We then deduce that $CH^g_{\mathrm{ind}}(J(C), 1)_s$ is uncountable for $4 \le s \le g$, as well as the statement for $k < g$, in the same way as in Theorem 1.2. $\qquad\square$

Consider the natural map $\tau_g : H'_g \to \mathcal{M}_g$. Since the map $C^{(2)} \times C^{(2)} \to J(C)$ given by $(\{a_1, a_2\}, \{b_1, b_2\}) \mapsto [a_1 + a_2 - (b_1 + b_2)]$ is surjective for any curve of genus $g \le 4$, by counting parameters we see that τ_g is dominant for $g = 3, 4$. For $g > 4$ a degeneration argument shows that the image has dimension $2g + 1$. Note that the (non-empty) fibres of τ_g are at least $3 = dim(Aut(\mathbb{P}^1))$ dimensional and the generic fibre of τ_3 is 4 dimensional.

Corollary 3.1. *For C a generic curve of genus $g = 3, 4$, $CH^g_{\mathrm{ind}}(J(C), 1)_g$ is uncountable.*

Proof. The statement for $g = 4$ follows from the theorem. For $g = 3$, the generic curve has a 1-parameter family of 4-configurations. Consider a family of genus 3 curves with special fibre a hyperelliptic curve as in the theorem. By varying the 4-configuration in the Jacobian of the generic fibre in the 1-parameter family, we obtain as specializations cycles of the form $K - K_t$ where now t varies in a 1-parameter family. By Theorem 1.2, it follows that the set of specializations is an uncountable subset of $CH^3_{\mathrm{ind}}(J(C_1), 1)_3$. Hence by Theorem 2.1 the 4-configurations also form an uncountable subset of $CH^3_{\mathrm{ind}}(J(C), 1)_3$. $\qquad\square$

4. HIGHER CHOW CYCLES ON SELF PRODUCTS OF A CURVE

4.1. Construction of the cycles. The indecomposability of the basic hyperelliptic cycle and the 4-configuration suggest that to construct indecomposable elements in $CH^g(J(C), 1)$ for more general Jacobians, we should consider more general divisors D on C, with $[D]$ of order 2 in $Pic(C)$. Let D be such a divisor, so $D = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$ for some $n > 0$ and distinct points $a_i, b_i \in C$, and there exists a rational function f on C with $div(f) = 2D$. Note that if $0 < g \le 2n$ then there always exist such divisors on a general curve, and if $g \ge 2n$ then there exists a $2g + 2n - 3$ dimensional family of curves of genus g which have such divisors.

Let E be the set of embeddings of C in C^{n+1} such that the composition of any $e \in E$ with the projection to any factor is either the identity map or is constant with image equal to one of the a_i's or b_i's. We have the following group actions on the set E.

(i) Σ_{n+1} acts by permuting the factors.
(ii) Σ_n acts by permuting the a_i's.
(iii) Σ_n acts by permuting the b_i's.

(iv) $\mathbb{Z}/2$ acts by switching a_i and b_i for all i.

Let f_e be the function f considered as a function on $e(C)$.

Proposition 4.1. *The set of higher Chow 1-cycles of the form $\sum_{e \in E} m_e(e(C) \otimes f_e)$, $e \in E$, $m_e \in \mathbb{Q}$ invariant under all the above group actions on E, is a positive dimensional vector space for all $n > 0$.*

Proof. By a partition of a positive integer n we shall mean a tuple of non-increasing positive integers $\alpha = (n_1, n_2, \ldots, n_r)$ such that $|\alpha| := \sum_{i=1}^{r} n_i = n$ and by a partition of zero we mean the empty tuple (). If α is as above and m is a positive integer, we let $\alpha + m$ be the partition of $n + m$ obtained by reordering $(n_1, n_2, \ldots, n_r, m)$.

The orbits of the action generated by all the above groups on E are in 1-1 correspondence with the set

$$\mathcal{E} = \left\{ \{\alpha, \beta\} \mid \alpha \text{ a partition of } i, \ \beta \text{ a partition of } j, \ i + j \leq n \right\}.$$

Here the elements of $\mathcal{E}$ (and $\mathcal{P}$ below) are viewed as unordered pairs.

The boundary of an element $\sum_{e \in E} m_e(e(C) \otimes f_e)$ is supported on points of C^{n+1}, each of whose coordinates is one of the a_i's or b_i's. The orbits of the set of all such points under the group action are in 1-1 correspondence with the set

$$\mathcal{P} = \left\{ \{\alpha, \beta\} | \alpha \text{ a partition of } i, \ \beta \text{ a partition of } j, i + j = n + 1 \right\} \setminus \left\{ \{(1, \ldots, 1), ()\} \right\}.$$

$\{(1, \ldots, 1), ()\}$ is not included because the number of a_i's and b_i's is n.

Let $\mathcal{R} \subset \mathcal{E} \times \mathcal{P}$ be the set

$$\bigcup_{\{\alpha, \beta\} \in \mathcal{E}} \left\{ (\{\alpha, \beta\}, \{\alpha, \beta + (n + 1 - |\alpha| - |\beta|)\}), (\{\alpha, \beta\}, \{\alpha + (n + 1 - |\alpha| - |\beta|), \beta\}) \right\}.$$

Consider the projection $p_2 : \mathcal{R} \to \mathcal{P}$ and observe that $|p_2^{-1}(\{\alpha, \beta\})| \geq 2$ for all $\{\alpha, \beta\} \in \mathcal{P}$ except for those of the form (i) $\alpha = (d, d, \ldots, d)$, $\beta = ()$, with $d > 1$ a divisor of $n + 1$ and (ii) $\alpha = \beta = (d, d, \ldots, d)$ with n odd and d a divisor of $(n + 1)/2$. For both these cases $|p_2^{-1}(\{\alpha, \beta\})| = 1$. Using the fact that for any integers d, m, if $m/2 < d < m$ then d cannot divide m, one checks that if $n > 3$, then $\left| \{\{\alpha, \beta\} \in \mathcal{P} \mid |p_2^{-1}(\{\alpha, \beta\})| = 1\} \right| \leq n$.

On the other hand, $|p_1^{-1}(\{\alpha, \beta\})| = 2$ for all $\{\alpha, \beta\} \in \mathcal{E}$ except for those with $\alpha = \beta$ or $\alpha = (1, 1, \ldots, 1)$, $|\alpha| = n$, and $\beta = ()$. For both these cases $|p_1^{-1}(\{\alpha, \beta\})| = 1$. If $\Pi(m)$ denotes the number of partitions of m, one sees that the number of such elements is $1 + \sum_{i=0}^{[n/2]} \Pi(i)$ which is greater than n for all $n > 5$.

Elements of $\mathcal{P}$ give us sufficient relations among the m_e's for the boundary of $\sum_{e \in E} m_e(e(C) \otimes f_e)$ to be zero, but if n is odd then elements of the form $\{\alpha, \alpha\}$, $|\alpha| = (n + 1)/2$, give trivial relations because of symmetry. Using this observation along with explicit computations for $1 \leq n \leq 5$, one sees that the number of relations is always strictly less than the number of variables i.e. $|\mathcal{E}|$, hence the space of invariant cycles is positive dimensional. $\qquad\qquad\square$

Let $V(C, f)$ be the space of all invariant higher Chow 1-cycles as in the proposition. By abuse of notation we shall also denote this space by $V(C, D)$. For $n = 1$ and 2, $V(C, D)$ is of rank 1, but it is of rank > 1 for all $n > 2$.

4.2. **Indecomposability of the product cycles via specialization.** Proposition 4.1 provides us with elements of $CH^{n+1}(C^{n+1}, 1)_{\mathbb{Q}}$. After pushforward by the natural map $C^{n+1} \to Pic^{n+1}(C)$ and translation by a divisor of degree $-(n + 1)$, we obtain elements of $CH^g(J(C), 1)_{\mathbb{Q}}$. We now outline a method which should

allow one to prove indecomposability of the image of a general element of $V(C, D)$ in $CH^g(J(C), 1)_{\mathbb{Q}}$, for C generic among curves of genus $g \geq n + 1 > 2$, having such divisors D.

Let $\mathcal{M}_g$ be the moduli space of genus g curves (pointed curves if $g = 1$) with level $2m$ structure for some $m \geq 3$ and let $\mathcal{C}_g$ be the universal family of curves over $\mathcal{M}_g$. Consider the map $\sigma : Sym^n(\mathcal{C}_g/\mathcal{M}_g) \times Sym^n(\mathcal{C}_g/\mathcal{M}_g) \to J(\mathcal{C}_g/\mathcal{M}_g)$ given on the fibres over $\mathcal{M}_g$ by $(\{a_1, a_2, \ldots, a_n\}, \{b_1, b_2, \ldots, b_n\}) \mapsto [\sum_n a_i - \sum_n b_i]$ and let $X_{g,n} = \sigma^{-1}(S_2)$, where S_2 is the union of sections of $J(\mathcal{C}_g/\mathcal{M}_g)$ corresponding to points of order 2 on each fibre. Let τ be the composite of σ with the natural map from $J(\mathcal{C}_g/\mathcal{M}_g)$ to $\mathcal{M}_g$.

Lemma 4.1. *Let $n > 1$ and $g > 0$. Then there exists an irreducible component $Y_{g,n}$ of $X_{g,n}$ with the following properties:*

(i) *$dim(Y_{g,n}) = 2g + 2n - 3$.*

(ii) *If $g \leq 2n$, then the map $\tau|_{Y_{g,n}} : Y_{g,n} \to \mathcal{M}_g$ is dominant and if $g \geq 2n$, then $dim(\tau(Y_{g,n})) = 2g + 2n - 3$.*

(iii) *In the fibre of $\tau|_{Y_{g,n}}$ over a point on $\mathcal{M}_g$ corresponding to a generic hyperelliptic curve C', there exist points $(\{w_1, t_1, t_2, \ldots, t_{n-1}\}, \{w_2, t_1, t_2, \ldots, t_{n-1}\})$, where w_1 and w_2 are distinct Weierstrass points on C', and $(t_1, t_2, \ldots, t_{n-1})$ is a generic point of C'^{n-1}.*

Outline of proof. The lemma can be proved by considering a suitable Hurwitz scheme as in section 3. We do not know the number of components if $n > 2$, but a suitable choice of the monodromy representation allows us to single out a component which gives rise to the desired specializations (c.f. the discussion before Lemma 3.1). For example, if $n = 3$ we would consider the representation in Σ_6 corresponding to the tuple

$$((23), (23), (45), (45), (12), (12), \ldots, (12), (12)(34)(56), (12)(34)(56)) .$$

We let $Y_{g,n}$ be the closure in $X_{g,n}$ of points of the form C, $\{a_1, a_2, \ldots, a_n\}$, $\{b_1, b_2, \ldots, b_n\}$, where C is a cover of $\mathbb{P}^1$ corresponding to a point on the chosen component of the Hurwitz scheme, and $\{a_1, a_2, \ldots, a_n\}$, $\{b_1, b_2, \ldots, b_n\}$ are the inverse images of the two points of $\mathbb{P}^1$ over which the cover is not simply ramified. Since the dimension of each component of the Hurwitz scheme is $2g + 2n$ and $dim(Aut(\mathcal{P}^1)) = 3$, it follows that $dim(Y_{g,n}) = 2g + 2n - 3$.

To prove the statement about the $\tau(Y_{g,n})$, we consider a degeneration of the cover such that three of the simply ramified points come together (generically) at a point. For instance, in the above example we would let p_5, p_6 and p_7 come together. If $g \geq 2$, the special fibre of the stable model of the degenerating family of genus g curves then consists of two smooth components, one of them a curve of genus $g-1$ which is a cover of $\mathbb{P}^1$ of the same type, and the other a generic elliptic curve. If $g = 1$, we obtain a nonconstant family of elliptic curves and so the statement is true in this case. The statement for $g = 2$ follows, since we may then assume that both components are generic elliptic curves.

If $2 < g \leq 2n$, we may assume using induction that the point of intersection of the two components is a generic point on the curve of genus $g-1$ (also generic). Thus $dim(\tau(Y_{g,n})) \geq dim(\tau(Y_{g-1,n})) + 3$, and so $\tau|_{Y_{g,n}}$ must be dominant. Finally, if $g > 2n$ we see that $dim(\tau(Y_{g,n})) \geq dim(\tau(Y_{g-1,n})) + 2$. Since $dim(Y_{g,n}) = 2g + 2n - 3$ and $dim(\tau(Y_{2n,n})) = 4n - 3$, it follows that we must have equality for all $g \geq 2n$. $\qquad\square$

The basic idea for the proof of indecomposability is now the same as that used for the 4-configuration i.e. we specialize to a hyperelliptic curve and then use the results of sections 1 and 2. In somewhat more detail, the argument is as follows. By Lemma 4.1 we may construct a family of smooth curves $\mathcal{C} \to S$ of genus g and a divisor $\mathcal{D} = \sum_n A_i - \sum_n B_i$ on $\mathcal{C}$, with S a smooth curve with a distinguished point s, such that the fibre over the generic point, (C, D) corresponds to a generic point of $Y_{g,n}$ and the fibre over s, (C', D'), corresponds to the special points of $Y_{g,n}$ in Lemma 4.1 (iii).

Let W be the subspace of $CH^g(J(C'), 1)$ obtained by mapping $V(C, D)$ to $CH^g(Pic^{n+1}(C), 1)_{\mathbb{Q}}$, specialization to $CH^g(Pic^{n+1}(C'), 1)_{\mathbb{Q}}$, and then translation by $-(n+1)[w_1]$. Note that specialization is always defined here, since we are free to modify f by a nonzero constant.

By the condition on D' it follows that f', the specialization of f must be a Weierstrass function on C' corresponding to the divisor $2w_1 - 2w_2$. The components of the support of elements of W are all images of the various diagonals in C'^{n+1}, and the function on each component is f'. Lemma 1.6 then allows us to assume that modulo decomposable elements each element of W is a sum of translates of the basic hyperelliptic cycle K. It is then clear that the results of sections 1 and 2 imply that if $g \geq n + 1$, the general element of $V(C, D)$ (with (C, D) corresponding to a generic point of $Y_{g,n}$) is indecomposable, provided that the following hypothesis is satisfied.

Hypothesis 4.1. *The image of W in $CH^g_{\mathrm{ind}}(J(C'), 1)_{\mathbb{Q}}$ is the vector space spanned by the cycle $K * ([t_1 - w_1] - e) * ([t_2 - w_1] - e) * \ldots * ([t_{n-1} - w_1] - e)$.*

Remark. The difficulty in verifying the hypothesis is purely combinatorial; everything can be computed explicitly for any given n. We have verified it using a computer for $2 \leq n \leq 6$. (In all these cases, the kernel of the map from W to $CH^g_{\mathrm{ind}}(J(C'), 1)_{\mathbb{Q}}$ consists of those cycles which do not contain the (small) diagonal of C^{n+1} in their support.) Using the dimension formula from Lemma 4.1, we see that $CH^g_{\mathrm{ind}}(J(C), 1)_{\mathbb{Q}}$ is nonzero for a generic curve with $3 \leq genus(C) \leq 12$. As in the case of the 4-configuration, Pontryagin product with zero cycles can be used to prove that for these cases $CH^g_{\mathrm{ind}}(J(C), 1)_{\mathbb{Q}}$ is in fact uncountable.

5. Higher analogues in lower genus

In the first part of the section we construct elements $F \in CH^3(J(C), 4-g)$, where the genus $g \leq 2$. The cycles F are natural generalizations of the 4-configuration and we expect that they should be strongly indecomposable in the sense that we explain below. In the second part we study a certain cycle $B \in CH^3(J(C), 2)$ where C is a bielliptic curve of genus 2. We show that B is (weakly) indecomposable using Lewis' criterion, which we show to hold by means of the same kind of proof as the one which was given in (1.3).

5.1. **Higher analogues of the 4-configuration in lower genus.** Bloch's groups $CH^p(X, n)$ can be described by means of chains built from those integral subvarieties of codimension p in $X \times (\mathbb{P}^1 - \{1\})^n$ which meet all the cubical faces over 0 or over ∞ properly. Consider a semistable degeneration B' of an abelian variety B with trivial extension class, then $B' = A \times \mathbb{C}^*$ with compactification $\bar{B}' = A \times \mathbb{P}^1$ and thus one may expect that $CH^m(A, n)$ should retain memory of the properties

of $CH^m(B, n-1)$. This was the guess that prompted our construction of the 4-configuration, which in a way can be seen as being the memory of cycles studied in [6]. In the same vein we describe now a unified procedure for building a series of cycles F in $CH^3(J(C), 4-g)$, $g = genus(C) \leq 3$. The first is the 4-configuration and each element is the memory of the preceding one.

Choose a class ϵ of order 2, consider a divisor $D := (a_1 + a_2) - (b_1 + b_2)$ on C with $class(D) = \epsilon$, and let f be a rational function with $div(f) = 2D$. We embed $C \to J(C)$ using the maps $\alpha_1(x) = [x - a_1]$, $\alpha_2(x) = [-x + a_2]$, $\alpha_3(x) = [x - a_1] + \epsilon$, $\alpha_4(x) = [-x + a_2] + \epsilon$, and define $C_i := \alpha_i(C)$. The useful property here is the fact that on C_1 the point which comes from a_i coincides with the point on C_2 from $a_{i\pm 1}$. The same happens for b_i with respect to the curves C_1 and C_4, and things stay the same for every other curve instead of C_1.

We start from genus 3, here we take $\beta_i = C \to J(C) \times \mathbb{P}^1$ to be the map $x \to (\alpha_i(x), f(x)^{s(i)})$, where $s(i)$ means ∓ 1, according to the parity of i.

The curves $K_i := \beta_i(C)$ in $J(C) \times \mathbb{P}^1$ meet properly the cubical faces, $\sum_{i=1}^{4} K_i$ is indeed a cycle and this is our element

$$F(f) := \sum_{i=1}^{4} K_i \in CH^3(J(C), 1).$$

Clearly $F(f)$ is equivalent to the the 4-configuration as it was previously defined.

To go higher in Chow groups we need at each step a new and convenient rational function. It can be found by imposing restrictions on D by means of conditions on f and this is the reason why we need to go down in genus.

In genus 2 we require $\epsilon = [w_1 - w_2]$, the distinguished ramification points of the Weierstrass double cover $h : C \to \mathbb{P}^1$. The condition is

$$(+) \qquad\qquad\qquad f(w_1) = f(w_2)$$

which is satisfied by a 1- dimensional family of a's and b's as above. Weil reciprocity and $(+)$ yield

$$(*) \qquad\qquad\qquad (h(a_1)h(a_2))^2 = (h(b_1)h(b_2))^2.$$

We may change h by a multiplicative constant so as to have

$$(**) \qquad\qquad 1 = (h(a_1)h(a_2))^2 = (h(b_1)h(b_2))^2.$$

Writing now $\beta_i(x) := (\alpha_i(x), f(x)^{s(i)}, h(x)^{2s(i)})$ then $K_i := \beta_i(C)$ is a curve in $J(C) \times \mathbb{P}^1 \times \mathbb{P}^1$ which meets properly the cubical faces. It is easy to check that

$$F(f, h) := \sum_{i=1}^{4} K_i \in CH^3(J(C), 2),$$

for instance one can see that the boundary is trivial over the point of C_1 which comes from w_1 by realizing that it coincides with the point on C_3 from w_2, and by using then the condition $f(w_1) = f(w_2)$.

In genus 1 having fixed the divisor D of class ϵ as before we choose further two rational functions h_1 and h_2 of degree 2 on E both ramified over 0 and ∞. Let $div(h_i) = 2(q_i' - q_i'')$. We require our choice to satisfy:

$$(1) \qquad\qquad [q_1' - q_1''] = [q_2' - q_2''] = \epsilon$$
$$(2) \qquad\qquad h_1(q_2') = h_1(q_2'') \quad and \quad h_2(q_1') = h_2(q_1'').$$

This can be done, see [4]. Here the conditions on f read

$$(+)\qquad\qquad\qquad f(q_i') = f(q_i'') \quad i = 1\,,\, 2\,.$$

As it was before this implies

$$(*)\qquad\qquad\qquad ((h_i(a_1)h_i(a_2))^2 = (h_i(b_1)h_i(b_2))^2\,.$$

We normalize the choice of the rational functions h_i by the request:

$$(**)\qquad\qquad 1 = ((h_i(a_1)h_i(a_2))^2 = (h_i(b_1)h_i(b_2))^2\,.$$

Using $\beta_i(x) := (\alpha_i(x), f(x)^{s(i)}, h_1(x)^{2s(i)}, h_2(x)^{2s(i)})$, we obtain

$$F(f, h_1, h_2) := \sum_{i=1}^{4} K_i \in CH^3(E, 3)\,.$$

The 4-configuration $F(x, f, h_1, h_2) \in CH^3(\mathbb{C}, 4)$ is constructed in the same way as $F(f, h_1, h_2)$ was. One thinks of $\mathbb{P}^1$ as having the structure of a degenerate Jacobian, given by the choice of a standard parameter x. The opposite map from $J(C)$ to $J(C)$ becomes $x \to x^{-1}$ and the sum of points corresponds to product of the coordinates. Translation by ϵ is here multiplication by -1. The condition $class(D) = \epsilon$ reads $a_1 a_2 = -b_1 b_2$ and $f := (x - a_1)^2 (x - a_2)^2 (x - b_1)^{-2} (x - b_2)^{-2}$. We may take explicitly: $h_1 = c_1(x - 1)^2(x + 1)^{-2}$, $h_2 = c_2(x - i)^2(x + i)^{-2}$. The requirements on f are

$$(+)\qquad\qquad\qquad f(1) = f(-1)\,,\quad f(i) = f(-i)$$

Our choice yields also : $f(0) = f(\infty)$. One has

$$(*)\qquad\qquad (h_i(a_1)h_i(a_2))^2 = (h_i(b_1)h_i(b_2))^2 \quad, i = 1, 2.$$

Choose and fix the constants c_1 and c_2 so that it is

$$(**)\qquad\qquad 1 = (h_i(a_1)h_i(a_2))^2 = (h_i(b_1)h_i(b_2))^2\,, i = 1, 2.$$

With this dictionary the maps $\alpha_i : \mathbb{P}^1 \to \mathbb{P}^1$ are defined as before (for instance $\alpha_1(t) = ta_1^{-1}$), here the range of α_i should be understood as the replacement of $J(C)$. In this way the maps $\beta : \mathbb{P}^1 \to (\mathbb{P}^1)^4$ are here $\beta_i(x) = (\alpha_i(x), f(x)^{s(i)}, h_1(x)^{2s(i)}, h_2(x)^{2s(i)})$. Setting again $K_i := \beta_i(\mathbb{P}^1)$ our cycle is then

$$F(x, f, h_1, h_2) := \sum_{i=1}^{4} K_i \in CH^3(\mathbb{C}^*, 4)\,.$$

Remark. One may define an element in $CH^a(X, b)$ to be strongly indecomposable if it is not in the image of $CH^{a-1}(X, b - 1) \otimes \mathbb{C}^\times$. We think that the higher 4-configurations are good candidates to strong indecomposability.

5.2. **The B-configuration.** Following the terminology of [12] we define the group of (weakly) decomposable cycles in $CH^p(X, 2) \simeq H^{p-2}(X, \mathcal{K}_p)$ to be :

$$CH_{\mathrm{dec}}^p(X, 2) := Im\big\{ K_2(\mathbb{C}) \otimes CH^{p-2}(X) \longrightarrow CH^p(X, 2) \big\}\,,$$

and thus the indecomposable group is

$$CH_{\mathrm{ind}}^k(X, 2) := CH^k(X, 2)\big/ CH_{\mathrm{dec}}^k(X, 2)\,.$$

It is known that translations on an elliptic curve E act trivially on $CH_{\mathrm{ind}}^2(E, 2)$, see [8, 3.10]. We show that on the contrary translations on a genus 2 bielliptic

Jacobian $J(G)$ operate non trivially on $CH^3_{\text{ind}}(J(G), 2)$. Our procedure is similar to the one which we have applied above in (1.3). We deal here with the *B-configuration* which is shown to be indecomposable by checking Lewis' condition on a cycle $\mathfrak{B}$ of $CH^3(J(G) \times G, 2)$.

S. Bloch in his seminal memoir [3] constructed certain elements $S_b \in \Gamma(E, \mathcal{K}_2))$ associated with a point b of finite order on an elliptic curve E. He proved that the real regulator image of S_b is not trivial for some curves with complex multiplication, and thus it is not trivial in general.

Consider a bielliptic curve G of genus 2 with associated map $\delta_G : G \to E_1$, and let $a : G \to J(G)$ be the Abel Jacobi map. In this way $Z(b) := a_* \delta_G^*(S_b)$ is a cycle in $CH^3(J(G), 2)$. Translation by an element $t \in Pic^0(G)$ maps $Z(b)$ to the cycle $Z_t(b)$, our aim is to prove

Theorem 5.1. *The B-configuration* $B(t) := Z_t(b) - Z(b)$ *is indecomposable for generic* t.

Note that $B(t)$ has trivial regulator image.

Proof. Consider the cycle $G \times Z(b) \in CH^2(G \times G, 2)$. The straight embedding $\sigma := id \times a : G \times G \to G \times J(G)$ maps it to $\mathfrak{S} := \sigma_*(G \times Z(b))$ in $CH^3(G \times J(G), 2)$. The twisted embedding $\tau(t, x) := (t, a(x) + (t - w_1)))$ gives instead $\mathfrak{T} := \tau_*(G \times Z(b))$, with section $\mathfrak{T}_*(t) = Z_t(b)$, and therefore $B(t)$ is the section at $t \in G$ of $\mathfrak{B} := \mathfrak{T} - \mathfrak{S}$.

We use the same type of notations as we did in part 1.2, in particular the holomorphic form ω_i^J comes from E_i. We need to consider also the forms $\nu := \bar{\omega}_1^J \wedge \omega_2^J$ on $J(G)$ and $\bar{\zeta}_2$ on G. The procedure of 1.2 gives here again: (i) $< R(\mathfrak{B}), \bar{\zeta}_2 \wedge \nu > \neq 0$. The Neron-Severi space of divisors with rational coefficients on $J(G)$ is isomorphic to the same space on the product of the two associated elliptic curves. On the gneric bielliptic Jacobian ν is orthogonal to the Neron-Severi group, because it is orthogonal to the elliptic curves. A look at the proof of the main theorem of [12] shows that this property of ν and (i) imply that the generic section $\mathfrak{B}_*(t)$ is indeed weakly indecomposable. $\square$

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